Module 5: Notes & Exercises

THE STRAIGHT LINE

The Gradient of a line: Values & Applications

- **Values of the gradient**
  - Consider these lines:
    - The gradients are:
      - **POSITIVE**
      - **NEGATIVE**
      - **ZERO**
      - **UNDEFINED**
  - Consider these lines:
    - The gradients are:
      - **POSITIVE**
      - **NEGATIVE**
      - **ZERO**
      - **UNDEFINED**

- **Parallel lines**
  - Parallel lines have **equal** gradients.
  - \( AB \parallel CD \iff m_{AB} = m_{CD} \)

- **Perpendicular lines**
  - In the figure, line \( \Theta \) is perpendicular to line \( \Theta \).
  - If the gradient of line \( \Theta \) is \( +\frac{2}{3} \),
    then the gradient of line \( \Theta \) will be \( -\frac{3}{2} \).
  - So: \( m_{\Theta} \times m_{\Theta} = \left( +\frac{2}{3} \right) \times \left( -\frac{3}{2} \right) = -1 \)
  - i.e. The product of the gradients of \( \perp \) lines is -1.
  - In the figure, line \( \Theta \perp \Theta \) \( \iff m_{\Theta} \times m_{\Theta} = -1 \)
    or \( m_{\Theta} = -\frac{1}{m_{\Theta}} \)

Collinear points

Three points A, B & C are collinear if the gradients of \( AB \) & \( AC \) are equal.

\( m_{AB} = m_{AC} \iff A, B \& C \) are collinear

Note: Point A is common

Also:
\( m_{AB} = m_{BC} \), where point B is common,
\( m_{AC} = m_{BC} \), where pt C is common

The Inclination of a line

Angles \( \alpha \) and \( \beta \) alongside are angles of inclination.

The Inclination of a line is the angle which it makes with the positive direction of the \( x \)-axis.

\( \alpha \) acute ... gradient of line is positive
\( \beta \) obtuse ... gradient of line is negative

The gradient of a line is also the tan of the \( \angle \) of inclination, i.e. \( m = \tan \alpha \) or \( \tan \beta \).

If \( a \) and \( b \) represent positive lengths, then:

\[ m = \frac{a}{b} \]
and
\[ \tan \alpha = \frac{\text{opp}}{\text{adj}} = \frac{a}{b} \]
and
\[ \tan \beta = -\frac{a}{b} \]
### Module 5: Notes & Exercises

**E.g. Acute angle \( \alpha \):**

\[
\tan \alpha = \frac{3}{2} \Rightarrow \alpha = \tan^{-1} \left( \frac{3}{2} \right) = 56,31^\circ
\]

- **The inclination of the line**

In the sketch alongside, the two parallel lines both have a positive gradient \( = \frac{3}{2} \).

\[\therefore \text{ They have the same inclination, } \alpha.\]

(The angles correspond.)

**Obtuse angle \( \beta \):**

\[
\text{The gradient } = \frac{3}{2} \quad \text{and} \quad \tan \beta = \frac{3}{2}
\]

- **The inclination of the line**

\[
\therefore \beta = 180^\circ - 56,31^\circ = 123,69^\circ
\]

- The gradient is NEGATIVE.

\[\therefore \beta \text{ is obtuse}\]

In the sketch alongside, the two parallel lines have a negative gradient \( = -\frac{3}{2} \).

\[\therefore \text{ They have the same inclination, } \beta.\]

(The angles correspond.)

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**The gradient of the line is the tan of the angle of inclination**

i.e. \( m = \tan \alpha \text{ or } tan \beta \) as in \( y = mx + c \)

**E.g.**

\[
\begin{align*}
\text{1. } & y = 3x + 1 \quad m = 3 \\
\therefore & \tan \alpha = 3 \\
\therefore & \alpha = 71,57^\circ
\end{align*}
\]

\[
\begin{align*}
\text{2. } & y = -x \\
\therefore & \tan \beta = -1 \\
\therefore & \beta = 180^\circ - 45^\circ = 135^\circ
\end{align*}
\]

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**Worked Example**

Write down (a) the gradient, and (b) the inclination of the following lines:

1. \( y \)
2. \( y \)
3. \( y \)
4. \( y \)

**Answers**

(a): \( \frac{2}{3} \)
(b): For both 1. & 2.: \( 33,69^\circ \)

For both 3. & 4.: \( 180^\circ - 68,2^\circ = 111,8^\circ \)

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**Note:**

From the gradient, we can calculate the inclination, or from the inclination, we can calculate the gradient.
Graphs in general

3 Basic facts about graphs in general

0: Axis intercepts
Every point on the y-axis has $x = 0$.
Every point on the x-axis has $y = 0$.

Θ: The equation
The equation of a graph is true for all points on the graph.
∴ The equation of the y-axis is $x = 0$;
& the equation of the x-axis is $y = 0$.

Θ: Types of graph
Different types/patterns are indicated by various equations.
(See the variations of the equation of a line below.)

Straight line graphs & their equations

**Standard forms**

There are 2 standard forms of the equation of a straight line:

- $y = mx + c$: where $m =$ the gradient & $c =$ the y-intercept
  
  When $m = 0$: $y = c$ . . . a line || x-axis
  
  When $c = 0$: $y = mx$ . . . a line through the origin
  
  Also: $x = k$ . . . a line || y-axis (see Case 1 on p. 5.6)

- $y - y_1 = m(x - x_1)$: where $m =$ the gradient
  & $(x_1; y_1)$ is a fixed point on the line.

  This standard form will be explained on page 5.9.

As with distance, midpoint and gradient, we will consider equations in the same 3 cases as on page 5.6.

**Case 1: Horizontal and Vertical lines**

- **Horizontal lines:**
  Have the equation $y = c$ (i.e. $y =$ a number)
  $\therefore$ The equation of a line above is: $y = 5$

- **Vertical lines:**
  have the equation $x = k$ (i.e. $x =$ a number)

**Case 2: Lines through the Origin**

The y-intercept would always be zero.
∴ $c = 0$

Substitute in $y = mx + c$:
∴ $y = mx$ — the standard form of lines through the origin

∴ The equation of the line above is: $y = \frac{2}{3}x$

**Case 3: “Other lines”**

When lines are not parallel to the axes or through the origin, we consider:

$y = mx + c$ or $y - y_1 = m(x - x_1)$

E.g. Substitute $m = 5$ and point $(1; 6)$ in:

$y = 5x + c$
∴ $6 = 5(1) + c \therefore y - y_1 = m(x - x_1)$
$\therefore y - 6 = 5(x - 1)$
∴ $y = 5x - 5$
∴ $1 = c$
∴ $y = 5x + 1$
Non-standard forms of the equation

e.g. (1) \(3x - 4y = 12\)
\[y = \frac{\frac{3}{4}x}{4} - 3\]

So, generally . . .

The Dual-intercept method . . .

determine the intercepts as follows.

The general form of the equation

\[ax + by + c = 0\] is the general form of the equation of a straight line. This form is useful when finding the axis-intercepts and/or the gradient.

e.g. \(2x + 3y + 6 = 0\)

Again, the 'dual-intercept' method is useful.

Finding the equation of a line . . .

1. Given \(m\) and \(c\):

1.1 Given: A line has a gradient of -2 and cuts the y-axis at 3.
Method: Substitute \(m = -2\) & \(c = 3\) in \(y = mx + c\).
Equation: \(y = -2x + 3\) ← The 'gradient-intercept' method

1.2 Given: A line \(\parallel\) to the line \(y = -x + 2\), passes through the point (0; 4)
Method: Substitute \(m = -1\) & \(c = 4\) in \(y = mx + c\).
Equation: \(y = -x + 4\) ←

2. Given \(m\) and a point:

2.1 Given: A line has a gradient of 3 and passes through the point (1; 6).
Method: Substitute \(m = 3\) & \((1; 7)\) in:

\[y = mx + c\] or \[y - y_1 = m(x - x_1)\]

\[\therefore 7 = (3)(1) + c\]
\[
\therefore 4 = c\]

\[\therefore y = 3x + 4\] ←
2.2 **Given:** A line passes through point (-2; 4) and is perpendicular to line \( y = 2x + 5 \).

**Method:** Substitute \( m = -\frac{1}{2} \) & (-2; 4) in:

\[
\begin{align*}
  y &= mx + c \\
  y - y_1 &= m(x - x_1)
\end{align*}
\]

\[
\begin{align*}
  \therefore 4 &= \left(-\frac{1}{2}\right)(-2) + c \\
  \therefore 4 &= 1 + c \\
  \therefore 3 &= c
\end{align*}
\]

**Equation:** \( y = -\frac{1}{2}x + 3 \) ☞

**3 Given 2 points:**

3.1 **Given:** A line passes through the points (-3; 1) and (4; -6).

**Method:**

- The gradient of the line, \( m = \frac{-6 - 1}{4 - (-3)} = \frac{-7}{7} = -1 \).
- Substitute \( m = -1 \) and a point, say (-3; 1):

\[
\begin{align*}
  y &= mx + c \\
  y - y_1 &= m(x - x_1)
\end{align*}
\]

\[
\begin{align*}
  \therefore 1 &= (-1)(-3) + c \\
  \therefore 1 &= 3 + c \\
  \therefore -2 &= c
\end{align*}
\]

**Equation:** \( y = -x - 2 \) ☞

3.2 **Given:** A line passes through points (-3; -2) and (-3; 5).

**No 'method' needed!**

NB: The \( x \)-coordinates are the same! . . . Draw a sketch!

- The line is parallel to the \( y \)-axis
- Calculating \( m \) is ‘not possible’ . . . The gradient is undefined!

**Equation:** \( x = -3 \) ☞

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**Facts about Points on Graphs and Points of Intersection**

**FACT 1**

If a point lies on a graph, the equation is true for its coordinates, i.e. the coordinates of the point SATISFY the equation . . . so substitute! and, conversely,

If a point (i.e. its coordinates) satisfies the equation of a graph (i.e. "makes it true"), then it lies on the graph. [See Q1 in Exercise 5.2 on p 5.11 in The Answer Series Mathematics Grade 11 3 in 1.]

**FACT 2**

The POINT(S) OF INTERSECTION of two graphs:

The coordinates of the point(s) of intersection of two graphs "obey the conditions" of both graphs, i.e. they SATISFY BOTH EQUATIONS SIMULTANEOUSLY.

They are found

- "algebraically" by solving the 2 equations (see below), or
- "graphically" by reading the coordinates from the graph.

**THESE 2 FACTS ARE CRUCIAL!!**

**Worked Example**

Find the points of intersection of the 2 lines . . .

\( y = x + 5 \) & \( y = -x + 1 \)

**Answer**

At the point of intersection, \( P \)

\[
\begin{align*}
  x + 5 &= -x + 1 \quad (both = y) \\
  2x &= -4 \\
  x &= -2 \\
  y &= x + 5 = 3 \quad or \quad y = -x + 1 = 3 \\
  \therefore \text{The point of intersection, } P \text{ is } (-2; 3)
\end{align*}
\]
REVISION OF FORMULAE

Consider two points \( A(x_1; y_1) \) and \( B(x_2; y_2) \):

**DISTANCE FORMULA**
\[
AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad \text{... Pythag.}
\]
\[
AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

**MIDPOINT**
\[
M = \left( \frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2} \right)
\]

**GRADIENT**
\[
m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}
\]

Also NOTE
\[
\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y_2 - y_1}{x_2 - x_1}
\]

\[\therefore \tan \theta = m\]

**FACTS ABOUT GRADIENTS**
- \( \parallel \) lines: equal gradients
- \( \perp \) lines: gradients neg. inv. of each other, i.e. \( m_1 \times m_2 = -1 \)
- For points A, B and C to be collinear: \( m_{AB} = m_{AC} = m_{BC} \)

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CIRCLES

**Circles with the origin as centre:**
True of any point \((x; y)\) on a circle with centre \((0; 0)\) and radius \(r\) is that:
\[
x^2 + y^2 = r^2
\]

**Circles with any given centre:**
True of any point \((x; y)\) on a circle with centre \((a; b)\) and radius \(r\) is that:
\[
(x - a)^2 + (y - b)^2 = r^2
\]

**A Tangent to a \( \odot \)**
is perpendicular to the radius of the \( \odot \) at the POINT of contact.

Therefore, to find the equation of a tangent we usually use "m and 1 point" in the straight line equation \( y - y_1 = m(x - x_1) \)

**Distance formula!**
(i.e. the Theorem of Pythagoras again)

**NOTE:**
A line and a circle (or parabola!) either
- "cut" (twice!) \[ \text{[secant]} \]
  (2 points in common)
o r
- "touch" (once!) \[ \text{[tangent]} \]
  (1 point in common)
o r
- don't cut or touch
  (no points in common)

and if we substitute \( y = mx + c \) into the equation of the \( \odot \)
there will either be 2 solutions, 1 solution or no solutions for \( x \),
resulting in one of the above scenarios

**Converting from**
general form \( Ax^2 + Bx + Cy^2 + Dy + E = 0 \)
to
standard form \( (x - a)^2 + (y - b)^2 = r^2 \)
(using completion of squares)

\[\text{e.g. } \quad x^2 - 6x + y^2 + 8y - 25 = 0 \]
\[\therefore x^2 - 6x + 9 + y^2 + 8y + 16 = 25 + 9 + 16 \]
\[\therefore (x - 3)^2 + (y + 4)^2 = 50 \]
i.e. a \( \odot \) with centre \((3; -4)\) & radius, \( r = \sqrt{50} \) (= \( \sqrt{50} \) units)

An interesting fact . . .
When 2 \( \odot \)'s touch, the distance between their centres = the sum of their radii (\& vice versa)
i.e. \( AB = r + R \)
\[\therefore \quad \text{for } \quad AB > r + R \quad \text{and } \quad AB < r + R \]

**FINAL ADVICE**
Use your common sense & ALWAYS DRAW A PICTURE !!!